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Arthur Cayley and the 'Gruppen Pest'

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The original contributions of Arthur Cayley to the Philosophical Magazine on group theory and his 'trees' are revisited and to some extend reinterpreted. Both topics were and are of enormous importance not only in physics (group theory, graph theory), but also in quite a few other disciplines as diverse as information technology or, for example, linguistics (trees, graph theory). In order to show that these two topics originally arose from interests in the theory of permutations also Cayley's 'Mousetrap' game is briefly mentioned.

Keywords: history of science

1. Introduction

Arthur Cayley (1821–1895) was not only one of the founders of the modern British school of pure mathematics, but also a very interesting contemporary of the Victorian epoque. After having spent his first eight years with his parents in St. Petersburg, Russia, where his father acted as a merchant, he was sent to a private school, after which he attended King's College School. Already there his extraordinary mathematical skills were noticed. Later on, in Trinity College, Cambridge, he acquired a MA degree and won a temporary fellowship. Because this fellowship was limited in time, he moved to London, where – working as a lawyer – he spent about 14 years during which he published a couple of hundred papers. Around 1863 he was appointed to the *Sadlerian*, the newly founded chair in pure mathematics in Cambridge, meaning that he had to give up his lucrative practice as a lawyer in London for the modest salary at this university. In 1875 Cayley was made an ordinary fellow of Trinity College, in 1883 he acted as the president of the *British Association for the Advancement of Science*.

Cayley was a frequent contributor to the Philosophical Magazine: between 1845 and 1865 about 32 publications [1–32] appeared in this journal, which show his broad interests in various fields of mathematics such as theory of quaternions [1–3], theory of permutations [4], theorems concerning factorials [5], the logarithm [10,12] probabilities [7,26], group theory [8,9,14], a generalization of the binomial theorem [6] or matters of geometry, see e.g. [32].

Since it is virtually impossible to comment on all these papers, in here emphasis shall be put first on his papers on group theory, since this part of mathematics influenced the physics in the twentieth century, in particular quantum mechanics, enormously. As is probably

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well-known the 'group theory mania' started already at an early stage of developments in quantum mechanics: The very first article on the connection between quantum mechanics and group theory most likely was Weyl's contribution [33] to the Zeitschrift für Physik in 1927, see also [34]. Shortly afterwards, the so-called '3 W's', namely the books by Weyl [35], Wigner [36] and van der Waerden [37] appeared more or less simultaneously and layed the ground for many more books on this topic to come (for a selection of these books or extended essays, see Refs. [38–54]. The use of group theory in quantum mechanics became so popular that most of those not aware of the usefulness of this approach started to speak of the 'Gruppen Pest' that 'haunts' the theoretical physics community.

The second topic to be picked up and revisited will be the famous 'Cayley trees' because of their importance in a variety of sciences. There is hardly any scientific field of interest in which 'Cayley trees' are not used as a tool of systematization at least implicitly. Manifestations of such 'trees' can be found in technological applications as well as e.g. in lingustics or sociology.

In the following quotations from Cayley's contributions to the Philosophical Magazine are displayed using emphasized letters, his equations there are denoted by the symbols *GT-n* or *Tree-n*.

2. Group theory

In his very first paper [8] on group theory Cayley starts out by defining what is meant by the concept of group, a formal scheme that at his time was very little known:

> A set of symbols,
>
> $$1, \alpha, \beta \ \ldots \tag{GT-1}$$
>
> all of them are different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to the set, is said to be a group*. It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group; or what is the same thing, that if the symbols of the group are multiplied together so as to form a table (Figure 1), thus; that as well each line as each column of the square will contain all the symbols $1, \alpha, \beta \ldots$ [8].

In the footnote (*),

> *The idea of a group as applied to permutations or substitutions is due to Galois, and the introduction of it may be considered as marking an epoch in the progress of the theory of algebraical equations [8].

reference is given to Galois[1], a French mathematician who for the very first time introduced the concept of groups. Of what kind the symbols $\alpha, \beta, \ldots$ possibly can be, namely permutations, a topic he dealt with in an earlier paper [4], or substitutions (although in his definition of substitutions permutations are already included), he already had stated in the introduction of his paper:

> Let θ be a symbol of operation, which may, if we please, have for its operand, not a single quantity x, but a system $(x, y \ldots)$, so that
>
> $$\theta(x, y, \ldots) = (x', y', \ldots). \tag{GT-2}$$
>
> where $x', y\prime \ldots$ are any functions whatever of $x, y \ldots$, it is not even necessary that $x', y' \ldots$ should be the same in number with $x, y \ldots$. In particular x', y', &*c* may represent a permutation

Further factors.

Nearer factors.	1	α	β	..
1	1	α	β	..
α	α	α^2	$\beta\alpha$	
β	β	$\alpha\beta$	β^2	
	:			

Figure 1. The very first group multiplication table, Cayley [8].

> of x, y, &c. θ is in this case what is termed a substitution; and if, … , the operand is a single quantity x, so that $\theta(x) = x' = fx$, θ is an ordinary functional symbol [8].

Since in Equation (GT-1) obviously the ‘number’ one occurs as the symbol for the identity element, rules for the addition and multiplication of substitutions were needed:

> Addition: It is not necessary (even if this could be done) to attach any meaning to a symbol such as $\theta \pm \phi$ to the symbol 0, nor consequently to an equation such as $\theta = 0$, or $\theta \pm \phi = 0$ [8].
> Identity: the symbol 1 will naturally denote an operation which … leaves the operand unaltered [8].
> Multiplication: a symbol $\theta\phi$ denotes the compound operation, the performance of which is equivalent to the performance, first of the operation ϕ, and then of the operation θ; $\theta\phi$ is of course in general different from $\phi\theta$. the symbols $\theta \ldots \phi$ are in general such that $\theta \cdot \phi\chi = \theta\phi \cdot \chi$, &c., so that $\theta\phi\chi, \theta\phi\chi\omega$, &c. have a definite signification independent of the particular mode of compounding the symbols. … It is easy to see that $\theta^0 = 1$, and that the index law $\theta^m.\theta^n = \theta^{m+n}$ holds for all positive or negative integer numbers, not excluding 0 [8].

In short, for the group in Equation (GT-1) the existence of an associative, non-distributive multiplication rule is assumed. Still missing was of course a statement about group closure:

> Suppose that the group
>
> $$1, \alpha, \beta \ldots$$
>
> contains n symbols, it may be shown that each of these symbols satisfies the equation
>
> $$\theta^n = 1; \tag{GT-3}$$
>
> so that a group may be considered as representing a system of roots of this symbolic binomial equation. It is, moreover, easy to show that if any symbol α of the group satisfies the equation $\theta^r = 1$, where r is less than n, then that r must be a submultiple of n; it follows that when n is a prime number, the group is of necessity of the form [8]

$$1, \alpha, \alpha^2 \ldots \alpha^{n-1}, (\alpha^n = 1).$$

Since the concept of group multiplication tables and his term 'symbolic binomial equation' were absolutely new, Cayley considered for $n = 4$ the group $G = (1, \alpha, \beta, \gamma)$ to point out the difference between the *symbolic equation* θ^n and the ordinary equation $x^n - 1 = 0$:

	1	α	β	γ
1	1	α	α^2	α^3
α	α	α^2	α^3	1
β	α^2	α^3	1	α
γ	α^3	1	α	α^2

$1, \alpha, \alpha^2, \alpha^3, (\alpha^4 = 1)$

ordinary equation

	1	α	β	γ
1	1	α	β	γ
α	α	β	γ	1
β	β	γ	1	α
γ	$\alpha\beta$	1	α	β

$1, \alpha, \beta, \alpha\beta, (\beta^2 = 1)$

symbolic equation

In the remainder of his first publication on groups Cayley discusses various kinds group tables, mostly arising from the conditions he places on certain elements of the group such as for example by demanding that $\alpha\beta = \beta\alpha$, or that a certain element, say $(\alpha\beta)^m = 1, m < n$, n being the number of elements in the group.

In Part II [9] of his treatise on groups, Cayley introduces yet another concept, namely

> Imagine the symbols [of operation] $L, M, N, \ldots$ such that L being any symbol of the system, $L^{-1}L$, $L^{-1}M$, $L^{-1}N$, is the group $1, \alpha, \beta$, ... Then, in the first place, M being any other symbol of the system, $M^{-1}L$, $M^{-1}M$, $M^{-1}N, \ldots$ will be the same group $1, \alpha, \beta$, ... In fact, the system $L, M, N \ldots$ maybe written $L, L\alpha, L\beta \ldots$; and if e, g. $M = L\alpha$, $N = L\beta$, then
>
> $$M^{-1}N = (L\alpha)^{-1}L\beta = \alpha^{-1}L^{-1}L\beta = \alpha^{-1}\beta, \qquad \text{(GT-4)}$$
>
> which belongs to the group $1, \alpha, \beta \ldots$ [9].

Since this statement reads very difficult indeed, it shall be cast into a more explicit form. Suppose Cayley' group is denoted by G and there exists a set $S(L)$ constructed by 'mapping' all elements in G via an operation L

$$S(L) = \{L, L\alpha, L\beta, \ldots\} \equiv \{L, M, N, \ldots\}. \qquad (1)$$

Taking now an arbitrary pair of elements $M, N \in S(L)$,

$$M = L\alpha, N = L\beta,$$

then by forming the inverse of M,

$$M^{-1} = (La)^{-1}, \qquad (2)$$

the product M^{-1} and N.

$$M^{-1}N = a^{-1}L^{-1}L\beta = a^{-1}\beta \in G, \qquad (3)$$

is an element of G. It should be noted that in Cayley's statement the existence of the inverse of L, $L^{-1}L = LL^{-1} = 1$, was tacitly assumed while the inverse of $a \in G$ was only introduced by 'construction', since up to now the only relation to the identity element in G is given in Equation (GT-3). At this stage one has to remember that for Cayley L can be either a permutational or a mapping operation, i.e. in general an operator.

Suppose now that all elements of the set $S(L)$ in Equation 2 are remapped via L^{-1} (multiplied from the right with L^{-1})

$$S(LL^{-1}) = \left\{LL^{-1}, L\alpha L^{-1}, L\beta L^{-1}, \ldots\right\}, \qquad (4)$$
$$\equiv \left\{LL^{-1}, ML^{-1}, NL^{-1}, \ldots\right\},$$

then it is easy to follow Cayley's subsequent statements:

> Next it may be shown that
>
> $$LL^{-1}, ML^{-1}, NL^{-1}, \ldots \qquad \text{(GT-5)}$$
>
> is a group, although not in general the same group as $1, \alpha, \beta \ldots$. In fact, writing $M = L\alpha$, $N = L\beta$, &c., the symbols just written down are
>
> $$LL^{-1}, L\alpha L^{-1}, L\beta L^{-1}, \ldots \qquad \text{(GT-6)}$$
>
> and we have e.g. $L\alpha L^{-1}$, $L\beta L^{-1}$, $L\alpha\beta L^{-1} = L\gamma L^{-1}$, where γ belongs to the group $1, \alpha, \beta$. The system $L, M, N, \ldots$ may be termed a group-holding system, or simply a holder; and with reference to the two groups to which it gives rise.

Reviewing again Cayleys Equations (GT-4)–(GT-6) in a more familiar theoretical language,[2] it is evident that in these equations he introduced group homomorphism $G \simeq G'$. In his subsequent statement

> Suppose that these groups are one and the same group $1, \alpha, \beta$.., *the system* L, M, N, is in this case termed a symmetrical holder, and in reference to the last-mentioned group is said to hold such group symmetrically. It is evident that the symmetrical holder L, M, N.. may be expressed indifferently and at pleasure in either of the two forms $L, L\alpha, L\beta$ and $L, \alpha L, \beta L$, i.e. we may say that the group is convertible with any symbol L of the holder, and that the group operating upon, or operated upon, by a symbol L of the holder produces the holder [9].

most likely by *symmetrical holder* group isomorphism is meant, i.e. $G \cong G'$.

> Suppose now that the group
>
> $$1, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \ldots \qquad \text{(GT-7)}$$
>
> can be divided into a series of symmetrical holders of the smaller group
>
> $$1, \alpha, \beta, \ldots \qquad \text{(GT-8)}$$
>
> The former group is said to be a multiple of the latter group, and the latter group to be a submultiple of the former group. Thus considering the two different forms of a group of six, and first the form
>
> $$1, \alpha, \alpha^2, \gamma\alpha, \gamma\alpha^2 \ (\alpha^3 = 1, \gamma^2 = 1, \alpha\gamma = \gamma\alpha) \qquad \text{(GT-9)}$$
>
> the group of six is a multiple of the group of three, $1, \alpha, \alpha^2$ (in fact, $1, \alpha, \alpha^2$ and $\gamma, \gamma\alpha, \gamma\alpha^2$ are each of them a symmetrical holder of the group $1, \alpha, \alpha^2$) … [9].

He cautiously adds:

> There would not, in a case such as the one in question, be any harm in speaking of the group of six as the product of the two groups $1, \alpha, \alpha^2$ and $1, \gamma$ but upon the whole it is, I think, better to dispense with the expression [9].

Perhaps it is appropriate to discuss Cayleys statements in (GT-7)–(GT-9) in terms of rather well-known groups of six elements. Suppose the abstract group G (in Cayley's notation) is defined by the group multiplication table in (6) and consider the following assignments (mappings) of the letters a, b and c to the permutation group S_3, and to the covering group of a regular triangle, the point group C_{3v}.

$$\left.\begin{array}{l} \{a,b,c\} \rightarrow (a)(b)(c) \rightleftarrows e \rightleftarrows 1, \\ \{b,a,c\} \rightarrow (ab) \rightleftarrows \sigma_3 \rightleftarrows \alpha, \\ \{c,b,a\} \rightarrow (ac) \rightleftarrows \sigma_2 \rightleftarrows \beta, \\ \{a,c,b\} \rightarrow (bc) \rightleftarrows \sigma_1 \rightleftarrows \gamma, \\ \{b,c,a\} \rightarrow (abc) \rightleftarrows C_3^+ \rightleftarrows \delta, \\ \{c,a,b\} \rightarrow (acb) \rightleftarrows C_3^- \rightleftarrows \epsilon, \end{array}\right\} . \tag{5}$$

The multiplication tables of S_3 and C_{3v} are given in (7) and (8). In Equation 5, the element $(ab) \in S_3$ refers to the operation that in the set $\{a,b,c\}$ a and b are interchanged, while (acb) corresponds to an operation in which first in $\{a,b,c\}$ b and c change position and then a and c. The symbols C_3^+ and C_3^- in (8) denote clockwise and anticlockwise rotations by 120°, and the σ_n refer to the mirror planes passing through the three corners of a regular triangle.

Group multiplication table of the abstract group $G = \{1, \alpha, \beta, \gamma, \delta, \epsilon\}$ (6)

G	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	1	ϵ	δ	γ	β
β	β	δ	1	ϵ	γ	α
γ	γ	ϵ	δ	1	β	α
δ	δ	δ	γ	α	ϵ	1
ϵ	ϵ	α	γ	β	1	δ

Group multiplication table of S_3 (7)

S_3	e	(ab)	(ac)	(bc)	(abc)	(acb)
e	e	(ab)	(ac)	(bc)	(abc)	(acb)
(ab)	(ab)	e	(acb)	(abc)	(bc)	(ac)
(ac)	(ac)	(abc)	e	(acb)	$ab)$	(ac)
(bc)	(bc)	(acb)	(abc)	e	(ac)	(ab)
(abc)	(abc)	(ac)	(bc)	(ab)	(acb)	e
(acb)	(acb)	(bc)	(ab)	(ac)	e	(abc)

Group multiplication table of C_{3v} (8)

C_{3v}	e	σ_3	σ_2	σ_1	C_3^+	C_3^-
e	e	σ_3	σ_2	σ_1	C_3^+	C_3^-
σ_3	σ_3	e	C_3^-	C_3^+	σ_1	σ_2
σ_2	σ_2	C_3^+	e	C_3^-	σ_1	σ_3
σ_1	σ_1	C_3^-	C_3^+	e	σ_2	σ_3
C_3^+	C_3^+	σ_2	σ_1	σ_3	C_3^-	e
C_3^-	C_3^-	σ_1	σ_3	σ_2	e	C_3^+

From these three tables one easily can see that the three groups G, S_3 and C_{3v} are isomorphic to each other

$$G \cong S_3 \cong C_{3v}, \tag{9}$$

as their multiplication tables show the same structure. Furthermore, as best being visualized from (8) they have two subgroups, namely $\{e\}$, a 'trivial' subgroup, and $C_3 = \left\{e, C_3^+, C_3^-\right\}$

$$\{e\} \subset C_3 \subset C_{3v}, \tag{10}$$

C_3 would in Cayleys notation be called a 'symmetrical holder' in C_{3v}. His very last statement, namely the one he rather would like to 'dispense', refers then to the fact that (without entering the concept of classes) C_{3v} can be viewed as being generated by C_3 and a left coset of C_3, for example

$$\begin{aligned} C_{3v} &= \{C_3, \sigma_1 C_3\}, \\ \sigma_1 C_3 &= \{\sigma_1, \sigma_2, \sigma_3\}. \end{aligned} \tag{11}$$

One has to appreciate that Cayley discovered in his two publications on 'group theory' not only the concept of group multiplication tables, but also that of subgroups and cosets, and of course of group homomorphism. According to the mathematical tools available at his time, the identity element in a group is defined via the 'symbolic binomial equation' in Equation (GT-3) as 'number' 1. It is worthwhile to mention that both papers on 'group theory' were written while Cayley acted as a lawyer in London. The address given at the very end of both papers is *2 Stone Buildings, W.C.*

It should be noted that now-a-days *Cayley's theorem* states that every group G is isomorphic to a subgroup of the symmetric group acting on G. It puts all groups on the same footing, by considering any group as a permutation group of some underlying set. As every group contains as 'trivial' subgroup the group itself, this theorem applies of course also to the above-discussed isomorphism between S_3 and C_{3v}. Clearly, for the permutation group it does not matter of what kind the symbols are in a particular set. In the case of S_3 one just as well can consider the sets $\{1, 2, 3\}$, $\{\Sigma, \Phi, \Omega\}$, or $\left\{\otimes, \boxtimes, \ \&\ \right\}$ instead of $\{a, b, c\}$.

Five years later, he published a third part [14] of his treatise on the theory of groups, in which, however, he discusses mainly groups of order 8 by putting different conditions on the group elements. No new concepts are added.

3. Group theory and quantum mechanics

If an operator commutes with the Hamiltonian then this operator is called a *constant of motion*. A constant of motion that leaves also the boundary conditions invariant is termed a *symmetry operator*. Clearly a set of symmetry operators that forms a group is then called the symmetry group of the Hamiltonian under investigation. Well-known cases are for example the point group symmetries of molecules, translational symmetry (space groups) in solid-state physics, the Poincaré and Lorentz groups in space-time theory, or the permutation group in order to distinguish between bosons and fermions. Quantum mechanical language seems to be almost unthinkable without group theoretical concepts. The number of group theory text books that appeared in the last few decades is indeed impressive and indicates the importance of a mathematical concept originally introduced by Galois and Cayley in the nineteenth century.

4. Analytical forms called trees

> I may mention, in conclusion, that I was led to the consideration of the foregoing theory of trees by Professor Sylvester's researches on the change of the independent variables in the differential calculus.
> 2 Stone Buildings.[W.C.] January 2, 1856.

The acknowledgement given in Cayley's first publication 'On the Theory of the Analytical Forms called Trees' [13] gives an account of his friendship with James Sylvester[3] with whom he spent many hours walking through the Lincoln's Inn Fields discussing mathematical problems. Most likely Cayley's theory of 'trees' was an outcome of one or more strollings with James Sylvester, since Sylvester is said to be (also) the inventor of the (mathematical) term 'graph'.

> The inspection of the[se] figures (in Figure 2) will at once show what is meant by the term in question [analytical forms called trees], and by the terms root, branches (which may be either main branches, intermediate branches, or free branches), and knots' (which may be either the root itself, or proper knots, or the extremities of the free branches) [13].

The invention of a concept of 'trees' is yet another example of Cayley's innovative thinking. Originally introduced to classify the action of operators P, Q, R, ... on an operand U, see the lower part of Figure 2, it quickly became a field on its own.

> If PU denotes the result of the operation P performed on U, then QPU denotes the result of the operation Q performed on PU...Now considering the expression QPU, it is easy to see that we may write
>
> $$QPU = (Q \times P)U + (QP)U, \qquad \text{(Tree-1)}$$
>
> where on the right-hand side $(Q \times P)$ and (QP) signify as follows: viz. $Q \times P$ denotes the mere algebraical product of Q and P, while $QP\ldots$ denotes the result of the operation Q performed upon P as operand [13].

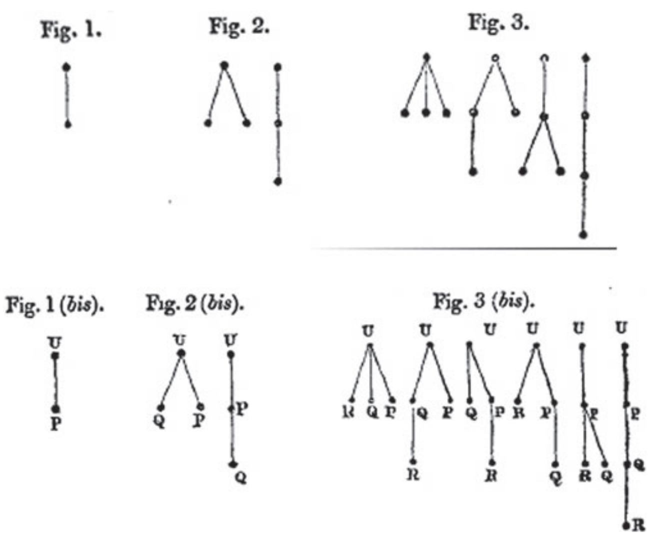

Figure 2. Roots, branches and knots [13].

In terms of these definitions Cayley's Fig. 2(bis) in Figure 2, for example, is very simple to understand: the left part refers to $(Q \times P)$, the right part to (QP) acting on U. Cayley's Fig. 3(bis) follows then directly from the meaning of Fig. 2(bis). Considering only the operator parts the following cases are graphically depicted in Fig. 3(bis),

$$\begin{array}{cc} (R \times Q \times P) & R \times (Q \to P) \\ (R \to Q) \times P & (R \times Q) \to P\,, \\ (R \times P) \times Q & R \to Q \to P \end{array}$$

where the arrow symbolizes 'acting on'. From Fig. 3(bis) also the concept of knots and branches becomes immediately clear.

Obviously having only three operators, the situation is still easy to envisage, for an arbitrary number, however, this virtually becomes impossible, or to use Cayley's own words: *the number of trees A_n with n branches is a very singular one* [13].

In his second note on 'Analytical Forms called Trees' [15], the main emphasis is put on counting the number of trees Φ_m that can be constructed for given number m of knots (terminal points), i.e. of interest are only the permutational properties of such trees. Since already his illustrations in Figure 2 indicated that this number might be obtained inspecting trees with less knots, in [15] he first illustrates (again) graphically all trees explicitly for $m \leqslant 4$, see the case of $m = 4$ in Figure 3, before giving a table for Φ_m, $m \leq 8$. It is indeed amazing to realize that for eight knots the number of trees is already 47293!

Cayley's table for the number of trees Φ_m corresponding to m knots [15]

m	1	2	3	4	5	6	7	8
Φ_m	1	1	3	13	75	541	4683	47293

Only then he proceeds in dealing with the general case of considering for a given number m of knots all trees with $n \leq m - 1$ knots:

$$\Phi_m = \Phi_1\Phi_{m-1} + \Phi_2\Phi_{m-2} + \Phi_3\Phi_{m-3} + \cdots \;\Phi_{m-1}\Phi_1. \qquad \text{(Tree-2)}$$

Using permutational arguments he finally succeeds to produce a closed formula for Φ_m [15]

$$\Phi_m = \frac{1.3.5.\ldots(2m-3)}{1.2.3.\ldots m}2^{m-1}. \qquad \text{(Tree-3)}$$

It should be mentioned that in deriving this formulae his switching between algebraical equations and permutational arguments is quite impressive.

5. Cayley's formula and graph theory

Now-a-days *Cayley's formula* in Equation (Tree-3) is part of graph theory [55]. Graphs can be used to model many types of relations and processes in physical, biological, social and information systems. In computer science, for example, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. Graph-theoretic methods, in various forms, have proven particularly useful in linguistics or in sociology as a way, for example, to measure the use of social networks, etc.

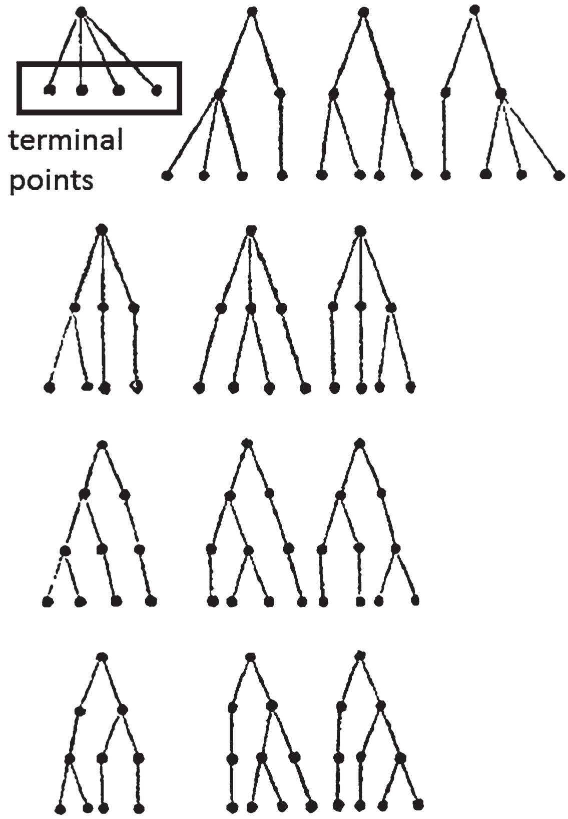

Figure 3. Possible trees with four knots (indicated as terminal points). From [15].

In mathematics, graphs are useful in geometry and certain parts of topology such as knot theory. Algebraic graph theory, by the way, has close links with group theory.

6. Cayley's 'mouse trap' game

In 1878 Cayley commented [56] in the *Quarterly Journal of Pure and Applied Mathematics* on his 'mousetrap' game [57], originally proposed in 1857:

> In the note 'A Problem in Permutation', Quarterly Mathematical Journal, t. I. , p. 79, I have spoken on the problem of permutations presented by this game [of Mousetrap].
>
> A set of cards – ace, two, three, &c., say up to thirteen – arranged (in any order) in a circle with their faces upwards; you begin at any card, and count one, two, three, &c., and if upon counting, suppose the number five, you arrive at the card five, the card is thrown out, and beginning with the next card, you count one, two, three, &c., throwing out if the case happen a new card as before, and so on until you have counted up to thirteen, without coming to a card which has been thrown out. The original question proposed was for any given number of cards to find the arrangement (if any) which would throw out all the cards in a given order; but (instead of this) we may consider all the different arrangements of the cards, and inquire how many of these

are in which all or any given smaller number of cards will be thrown out; and (in the several cases) in what order the cards are thrown out.

and quoted for four cards the respective results. More than a hundred years later it was observed that 'the game of Mousetrap, a problem in permutations, first introduced by Arthur Cayley in 1857 . . . has been largely unexamined since' [58]. It obviously needed computers to arrive at somewhat satisfactory results. It is therefore hardly surprising to find different answers for more than four cards in different sources.

7. Résumé

Cayley's 'Mousetrap' was mainly mentioned to indicate his long-lasting interests in the theory of permutations. Over a period of about 30 years, from his notes on permutations [4] in 1849, on determinants [17] in 1861, to the combinatorical problem posed by the 'Mousetrap', he dealt with different manifestations or applications of permutation theory. Considering that in the nineteenth century the permutation group was the only group of interest in group theory and his 'trees' also led to a problem to be solved using permutations, these two topics fit perfectly in this pattern. Perhaps his friendship with James Sylvester inspired both to contribute significantly to a new school of British mathematics, to introduce absolutely new mathematical concepts.

It was the purpose of this comment to revisit some of Cayley's important contributions to the Philosophical Magazine, which in the nineteenth century was also one of the leading journals in pure and applied mathematics. Obviously much more can be said about Cayley's other contributions to pure mathematics, just as well as on the two topics chosen, group and graph theory. Deliberately as little as possible formal language was used in order to stress or 'translate' the original formulations, to make this comment readable to a broader audience, in particular, since Victorian scientific language is not always easy to read. It remains to be said that Cayley's unconventional approaches were and are indeed admirable.

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes

1. Évariste Galois (1811–1832) was a French mathematician. While still in his teens, he was able to determine a necessary and sufficient condition for a polynomial to be solvable by radicals, thereby solving a problem standing for 350 years. His work laid the foundations for Galois theory and group theory, two major branches of abstract algebra, and the subfield of Galois connections. He died at age 20 from wounds suffered in a duel, http://en.wikipedia.org/wiki/Évariste.
2. Let $f : G \to G'$ be a homomorphic mapping between groups G and G', $f(G)$ being called the homomorphic mapping of G, then $f(G)$ is a subgroup of G', $G \subset G'$, because

   (1) the identity element $f(e)$, e being the identity element in G, $eg = g$, is the identity element G', since $f(eg) = f(e)f(g)$, see his definition of identity.
   (2) For $g^{-1}g = e \in G$ follows that $f(g^{-1}g) = f(g^{-1})f(g)$, i.e. $f(g^{-1}) = [f(g)]^{-1}$, see (GT-3).
   (3) If $f(g)$ and $f(g') \in f(G)$, then $[f(g)]^{-1}f(g') \in f(G)$, since $[f(g)]^{-1} f(g') = f(g^{-1})f(g') = f(g'') \in f(G)$.

3. James Joseph Sylvester FRS (1814–1897) was an English mathematician. He made fundamental contributions to matrix theory, invariant theory, number theory, partition theory and combinatorics. He played a leadership role in American mathematics in the later half of the 19th century as a professor at the Johns Hopkins University and as founder of the American Journal of Mathematics. At his death, he was professor at Oxford. Cambridge University denied him for 35 years his B.A. and M.A., because of being Jewish he was not willing to accept the Thirty-Nine Articles of the Church of England. There are quite a few contributions of James Sylvester to be found in the Philosophical Magazine Archives.

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