Philosophical Magazine, 2014
Vol. 94, No. 12, 1263–1276, http://dx.doi.org/10.1080/14786435.2014.887860

Hamilton and the square root of minus one

Peter Weinberger[a,b,*]

[a] *Center for Computational Nanoscience, Seilerstätte 10/22, A1010 Vienna, Austria;* [b] *Department of Physics, New York University, 4 Washington Place, New York, NY 10003, USA*

(*Received 29 August 2013; accepted 20 January 2014*)

Quaternions, objects consisting of a scalar and a vector, sound like a mysterious concept from the past. In the nineteenth century, the theory of quaternions was praised as one of the most brilliant achievements in mathematical physics. The originator of this theory, Hamilton, surely one of the greatest scientists in that area, spent about 18 years in discussing all kinds of algebraic and geometric properties of quaternions. His research was communicated to the Philosophical Magazine in three series of papers comprising a total of 29 contributions. In this commentary, these three series of papers are revisited concentrating primarily on the algebraic properties of quaternions.

Keywords: geometry; modelling; statistical physics; structure; theoretical; transformations

1. Introduction

In 1844, William Rowan Hamilton (1805–1865) published his first paper in the Philosophical Magazine on a topic, originally meant to deal with new forms of 'imaginaries', that, however, turned over the years into an extensive theory of quaternions. In three series of papers, [1–21,23–27], entitled by *On quaternions: or on a new system of imaginaries in algebra* (1844–1850), *On continued fractions in quaternions* (1852–1853) and *Some extension of quaternions* (1854–1855), he discussed a mathematical concept[1] dealing with the algebraic and geometric properties of objects consisting of a scalar and a vector. In fact, it was only in 1862 when he communicated a last comment [29] on quaternions.

In the following, these three series of papers are revisited. Since Hamilton's publications are not easy to follow and since perhaps quaternions are still a bit mysterious, 'intermissions' are provided to summarize the theoretical progress Hamilton made in discussing quaternions. In particular, it will also be pointed out that why only algebraic properties are considered, leaving Hamilton's geometric interpretations essentially aside.

2. Imaginary units and rotations

It seems incredible that Hamilton did not bother at all to write an introduction in his very first paper, he abruptly starts with the following definition[2]:

*Email: peter@pwein.at

Let an expression of the form

$$Q = w + ix + jy + kz \qquad \text{(Hamilton 1)}$$

be called a quaternion, when w, x, y, z, which we shall call the four ***constituents****, of the* ***quaternion***[3] *Q, denote any real quantities, positive or negative or null, but i, j, k are symbols of three imaginary quantities, which we shall call* ***imaginary units****, and shall suppose to be unconnected by any linear relation with each other ... .* [1]
and immediately adds the identity-, addition and multiplication rules of such quantities

$$Q' = w' + ix' + jy' + kz', \qquad \text{(Hamilton 2)}$$
$$Q = Q' : w = w', x = x', y = y', z = z', \qquad \text{(Hamilton 3)}$$
$$Q \pm Q' = w \pm w' + i(x \pm x') + j(y \pm y') + k(z \pm z'), \qquad \text{(Hamilton 4)}$$

$$\begin{aligned} QQ' = ww' &+ iwx' + jwy' + kwz' \\ &+ ixw' + i^2xx' + ijxy' + ikxz' \\ &+ jyw' + jiyx' + j^2yy' + jkyz' \\ &+ kzw' + kizx' + kjzy' + k^2zz'. \end{aligned} \qquad \text{(Hamilton 5)}$$

Clearly, before being able to prove that the product QQ' is also a quaternion, conditions on the nine occurring products $i^2, ij, ik, ji, \ldots$ have to be made. In order to proceed, Hamilton makes another incredible statement:

Considerations[4]*, which it might occupy too much space to give an account of on the present occasion, have led the writer to adopt the following system of values or expressions for these nine squares or products*

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, && \text{(A.)} \\ &ij = k, \quad jk = i, \quad ki = j, && \text{(B.)} \\ &ji = -k, \quad kj = -i, \quad ik = -j, && \text{(C.)} \end{aligned} \qquad \text{(Hamilton 6)}$$

though it must, at first sight, seem strange and almost unallowable, to define that the product of two imaginary factors in one order differs (in sign) from the product of the same factors in the opposite order ($ji = -ij$). It will, however, it is hoped, be allowed, that in entering on the discussion of a new system of imaginaries, it may be found necessary or convenient to surrender some of the expectations suggested by the previous study of products of real quantities, or even of expressions of the form $x + iy$, in which $i^2 = -1$. And whether the choice of the system of definitional equations, (A.), (B.), (C.) has been a judicious, or at least a happy one, will probably, be judged by the event, that is, by trying whether those equations conduct to results of sufficient consistency and elegance [1].

Perhaps the last sentence in this quotation was the incentive for Hamilton to study the properties of quaternions over a period of 18 years, producing a total of 29 (!) communications on this topic to the Philosophical Magazine.

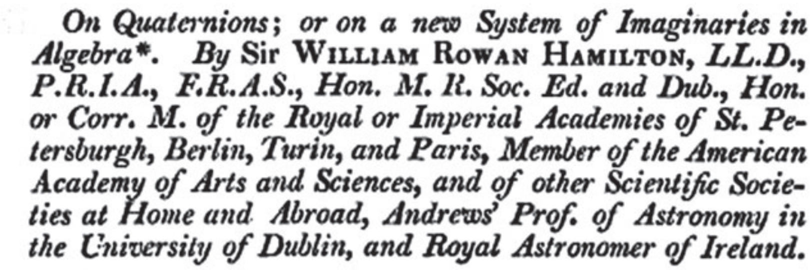
On Quaternions; or on a new System of Imaginaries in Algebra. By* Sir WILLIAM ROWAN HAMILTON, *LL.D., P.R.I.A., F.R.A.S., Hon. M. R. Soc. Ed. and Dub., Hon. or Corr. M. of the Royal or Imperial Academies of St. Petersburgh, Berlin, Turin, and Paris, Member of the American Academy of Arts and Sciences, and of other Scientific Societies at Home and Abroad, Andrews' Prof. of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

Figure 1. Title of his very first contribution on quaternions in the Philosophical Magazine [1]. Very typical for the style of publications in the nineteenth century, all his honours and academic memberships are listed together with his name.

Making use of the relations in Equation (Hamilton 6) in Equation (Hamilton 5), $Q'' = w'' + ix'' + jy'' + kz''$,

$$\begin{aligned} w'' &= ww' - xx' - yy' - zz', \\ x'' &= wx' + xw' + yz' - zy', \\ y'' &= wy' + yw' + zx' - xz', \qquad \text{(D.)} \\ z'' &= wz' + zw' + xy' - yx', \end{aligned} \tag{Hamilton 7}$$

and using trigonometric expressions for the ***constituents*** of Q,

$$\begin{aligned} w &= \mu \cos\theta, \\ x &= \mu \sin\theta \cos\phi, \\ y &= \mu \sin\theta \sin\phi \cos\psi, \\ z &= \mu \sin\theta \sin\phi \sin\psi, \end{aligned} \tag{Hamilton 8}$$

and in a similar manner $\mu', \theta', \phi', \psi'$ and $\mu'', \theta'', \phi'', \psi''$ for Q' and Q'', Hamilton found that

$$\begin{aligned} &\mu'' = \mu\mu', \\ &w''^2 + x''^2 + y''^2 + y''^2 = (w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2)\,, \end{aligned} \tag{Hamilton 9}$$

and thus according to him a simple geometrical interpretation is at hand. *Let xyz be considered as the three rectangular co-ordinates of a point in space, of which the radius vector is $\eta \sin\theta$,the longitude ψ, and the co-latitude ϕ; and let these three latter quantities be called also the radius vector, the longitude and the co-latitude of the quaternion Q; while θshall be called the amplitude of that quaternion. Let R be the point where the radius vector, prolonged if necessary, intersects the spheric surface described about the origin of co-ordinates with a radius = unity, so that ϕ is the co-latitude and ψ is the longitude of R; and let this point R be called the representative point of the quaternion Q.* [1] For an illustration of Hamilton's geometrical concept, see Figure 2.

It is interesting to note that many subsequent papers were already in principle contained in very first ones, in particular, as far as Hamilton's intentions were concerned to prove that a set of quaternions does form an algebra and, in particular, that quaternions are a perfect tool to describe rotations.

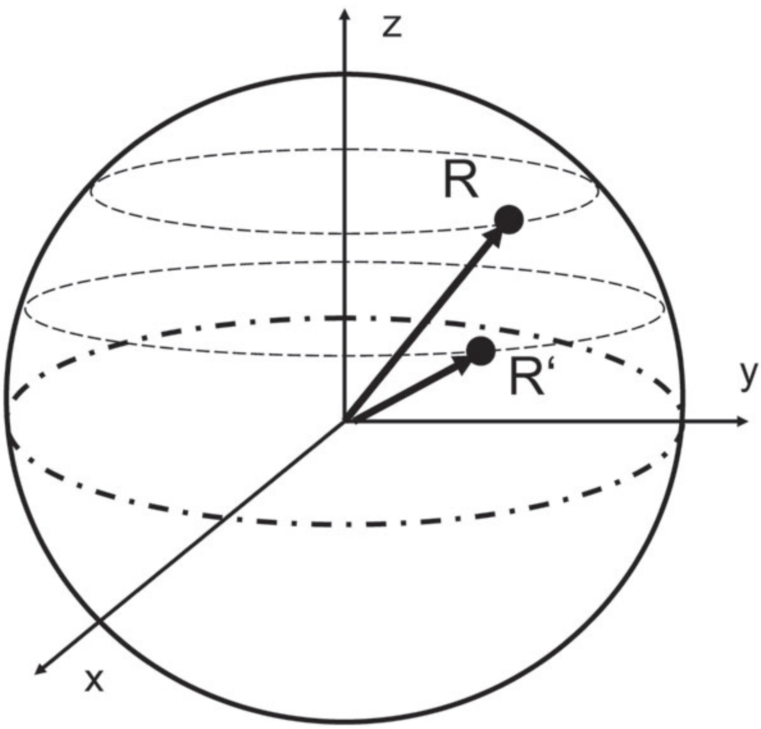

Figure 2. Sphere of unit radius with Hamilton's points R and R' indicated, since without loss of generality μ in Equation (Hamilton 7) can be taken to be 1.

Let R' and R'' *be, in like manner, the representative points of* Q' and Q''; *then … in the spherical triangle* $RR'R''$ *formed by the representative points of the two factors and the product (in any multiplication of two quaternions), the angles are respectively equal to the amplitudes of those two factors, and the supplement of the amplitude of the product (to two right angles); in such a manner that we have the three equations:*

$$R = \theta, \quad R' = \theta', \quad R'' = \pi - \theta'. \qquad \text{(Hamilton 10)}$$

Hamilton realized of course immediately *that they* [the above three equations] *leave undetermined the hemisphere to which the point* R'' *belongs, or the side of the arc* RR' *on which that product-point* R'' *falls, after the factor-points* R *and* R', *and the amplitudes* θ *and* θ' *have been assigned* …. and proposed a rule of how to perform the multiplication of quaternions: *it is not difficult to perceive that the product-point* R'' *is always to be taken to the right, or always to the left of the multiplicand-point* R', *with respect to the multiplier-point* R … [1].

The last sentence in Hamilton's first publication on quaternions: *This rule of rotation, combined with the law of the moduli and with the theorem of the spherical triangle, completes the transformed system of conditions, connecting the product of any two quaternions with the factors, and with their order* [1] was not really the end of his investigations: a few month later, in the continuation [2] of this paper, he made a few more interesting discoveries, but also got a little bit trapped by his obsession to usage of spherical trigonometry.

The first observation he made concerned a limiting process, namely when the spherical triangle $RR'R''$ becomes indefinitely small and the coordinates of R and R' approach asymptotically (1,0,0). Then,

$$(\cos R + i \sin R)\left(\cos R' + i \sin R'\right) = \cos(R + R') + i \sin(R + R'). \quad \text{(Hamilton 11)}$$

To this equation, he added that it *has so many important applications in the usual theory of imaginaries.* But: *In that theory there are only two different square roots of negative unity, and they differ only in their signs* [2].

However, *in the theory of quaternions, in order that the square of* $w + ix + jy + kz$ *should be equal to* -1, *it is necessary and sufficient that we should have*

$$w = 0, \quad x^2 + y^2 + z^2 = 1, \quad \text{(Hamilton 12)}$$

for, in general, by the expressions (D.) [Equation (Hamilton 7)] *of this paper for the constituents of a product, or by the definitions (A.), (B.), (C.), we have*

$$(w + ix + jy + kz)^2 = w^2 - x^2 - y^2 - z^2 + 2w(ix + jy + kz). \quad \text{(Hamilton 13)}$$

There are, therefore, in this theory, infinitely many different square roots of negative one, which have all one common modulus = 1, and one common amplitude = $\pi/2$, *being all of the form*

$$\sqrt{-1} = i \cos\phi + j \sin\phi \cos\psi + k \sin\phi \sin\psi, \quad \text{(Hamilton 14)}$$

but which admit of all varieties of directional co-ordinates, that is to say, co-latitude and longitude, since ϕ *and* ψ *are arbitrary; and we may call them all* ***imaginary units***, *as well as the three original imaginaries,* i, j, k, *from which they are derived.*

3. Intermission I

Before going ahead with the progress Hamilton made after 1844 in dealing with quaternions, it seems to be absolutely necessary to cast his initial ideas in a for present readers perhaps more transparent form.

(1) The relations in Equation (Hamilton 6) should be read as a multiplication table,

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

since by using his relation (A.) in relation (B.) would yield $\sqrt{-1}\sqrt{-1} = i^2 = -1 \neq k = \sqrt{-1}$.

(2) To understand his relations (B.) and (C.) [in Equation (Hamilton 6)] in an easy way, take the square of a quaternion (a quaternion multiplied by itself) and use without loss of generality $w = 0$ (rotations are explained in terms a sphere of unit radius anyhow). Remembering that x, y and z are coordinates, i.e. are expressed in numbers, which of course are commutative, then, in order to obtain a form that

contains only the squares of x, y and z, the conditions expressed by Hamilton as (B.) and (C.) have to be made.

$$w = 0 :$$
$$Q^2 = i^2x^2 + j^2y^2 + k^2z^2 + \underbrace{ijxy + \ jiyx}_{=0} + \underbrace{ikxz + kizx}_{=0} + \underbrace{jkyz + kjzy}_{=0}, \qquad \text{(Hamilton 5)}$$
$$Q^2 = -(x^2 + y^2 + z^2) = -1. \qquad \text{(Hamilton 12a)}$$

(3) The letters i, j and k Hamilton introduced are still very common to denote the (rectangular) ‘unit vectors’ for a three-dimensional vector. From Equation (Hamilton 12a) it is clear that a ***pure quaternion***, $Q = ix + jy + kz$, i.e. a quaternion corresponding to $w = 0$, is not a polar vector, but more like an axial vector, since only $Q^4 = 1$, in fact a pure quaternion is a so-called binary rotation, see p. 201 of Ref. [30].

(4) In terms of considering quaternions as elements of an algebra, Hamilton’s ‘parametrization’ in Equation (Hamilton 6) is of course perfectly logical, it turned out, however, that an ‘interpretation’ in terms of rotations is misleading as was already noted by Cayley in 1845 [31] and again in 1848 [32]. For this very reason, as indicated in the introduction, in the following, mostly the ‘algebraic’ aspects in Hamilton’s series of papers will be revisited: the concept of rotations is left to a proper theory of rotations[5] such as presented in Ref. [30].

4. Further properties of quaternions

4.1. *Multiple products*

It should be noted that from the above multiplication table, immediately the properties of (multiple) products can be read off, since for example

$$i(jk) = i(i) = -1.$$

Hence, whatever three quaternions may be denoted by Q, Q', Q'', we have the equation

$$Q.Q'Q'' = QQ'.Q'', \qquad \text{(Hamilton 15)}$$

and in like manner, for any four quaternions [2]

$$Q.Q'Q''Q''' = QQ'.Q''Q''' = QQ'Q''.Q'''. \qquad \text{(Hamilton 16)}$$

4.2. *Distributive law*

… quaternion-multiplication possesses … the distributive character of multiplication commonly so called, or in symbols, [2]

$$Q(Q' + Q'') = QQ' + QQ'',$$
$$(Q + Q')Q'' = QQ'' + Q'Q''. \qquad \text{(Hamilton 17)}$$

4.3. *Associative law*

… for quaternions as for ordinary factors, we may in general associate the factors among themselves, by groups, in any manner which does not alter their order; for example,

$$Q.Q'Q''.Q^{IV} = QQ'.Q''Q'''Q^{IV}; \qquad \text{(Hamilton 18)}$$

this, therefore, which may be called the associative character of the operation, is (like the distributive character) common to the multiplication of quaternions and to that of ordinary quantities, real or imaginary [2].

5. Scalars, vectors and tensors

In an attempt to define the inverse of a rotation and the so-called identity operation, Hamilton returned to geometrical aspects of quaternions in the second continuation [3] of his original article [1], namely to rotations. However, as was already said, since even for pure quaternions, see Equation (Hamilton 12a), only $Q^4 = 1$, this attempt ended in rather messy arguments.

In 1846, in his third continuation [4] of [1], he decided to give names to the parts of a quaternion, terms which became standard since then, although being used nowadays mostly in a different context.

5.1. *Scalars*

The separation of the real and imaginary parts of a quaternion is an operation of such frequent occurrence, and may be regarded as being so fundamental in this theory, that it is convenient to introduce symbols which shall denote concisely the two separate results of this operation. The algebraically real part may receive, according to the question in which it occurs, all values contained on the one scale of progression of number from negative to positive infinity; we shall call it therefore the ***scalar part****, or simply the* ***scalar*** *of the quaternion, and shall form its symbol by prefixing, to the symbol of the quaternion, the characteristic* ***Scal.****, or simply* ***S****., where no confusion seems likely to arise from using this last abbreviation* [4].

5.2. *Vectors*

On the other hand, the algebraically imaginary part, being geometrically constructed by a straight line, or radius vector, which has, in general, for each determined quaternion, a determined length and determined direction in space, may be called the ***vector part****, or simply the* ***vector*** *of the quaternion ; and may be denoted by prefixing the characteristic* ***Vect.****, or* ***V*** [4].

$$Q = Scal.Q + Vec.Q, \qquad \text{(Hamilton 19)}$$
$$Q = SQ + VQ.$$

By detaching the characteristics of operation from the signs of the operands, we may establish for this notation, the general formulae

$$1 = S + V; \quad 1 - S = V; \quad 1 - V = S; \qquad \text{(Hamilton 20)}$$
$$S.S = S; \quad S.V = 0; \quad V.S = 0; \quad V.V = V;$$

and may write

$$(S + V)^n = 1, \qquad \text{(Hamilton 21)}$$

if n be any positive whole number [4].

5.3. *Tensors*

Another general decomposition of a quaternion, into factors instead of summands, may be obtained in the following way: Since, the square of a scalar is always positive, while the square of a vector is always negative, the algebraical excess of the former over the latter square is always a positive number; if then we make

$$(TQ)^2 = (SQ)^2 - (VQ)^2, \qquad \text{(Hamilton 22)}$$

and if we suppose TQ to be always a real and positive or absolute number, which we may call the ***tensor*** *of the quaternion Q, we shall not thereby diminish the generality of that quaternion. … as some justification of the use of this word, or at least as some assistance to the memory, that it enables us to say that the tensor of a pure imaginary, or vector, is the number expressing the length or linear extension of the straight line by which that algebraical imaginary is geometrically constructed* [4].

5.4. *Versors*

If such an imaginary [a pure imaginary, or vector] *be divided by its own tensor, the quotient is an imaginary or vector unit, which marks the direction of the constructing line, or the region of space towards which that line is turned; hence, and for other reasons, we propose to call this quotient the* ***versor*** *of the pure imaginary, and generally to say that a quaternion is the product of its own tensor and versor factors, or to write*

$$Q = TQ \cdot UQ, \qquad \text{(Hamilton 23)}$$

using U for the characteristic of versor, as T for that of tensor. This is the other general decomposition of a quaternion, referred to at the beginning of the present article; and in the same notation we have

$$T \cdot TQ = TQ; \quad T \cdot UQ = 1; \quad U \cdot TQ = 1; \quad U \cdot UQ = UQ; \qquad \text{(Hamilton 24)}$$

so that the tensor of a versor, or the versor of a tensor, is unity, as it was seen that the scalar of a vector, or the vector of a scalar, is zero [4].

5.5. *Conjugation*

If we call two quaternions conjugate when they have the same scalar part, but have opposite vector parts, then because by the last article,

$$(TQ) = (SQ + VQ)(SQ - VQ), \qquad \text{(Hamilton 25)}$$

we may say that the product of any two conjugate quaternions, $SQ + VQ$ and $SQ - VQ$, *is equal to the square of their common tensor,* TQ*; from which it follows that* ***conjugate*** versors *are the reciprocals of each other, one quaternion being called the reciprocal of another when their product is positive unity* [4].

6. Intermission II

Quite clearly the above definitions introduced by Hamilton need to be critically interpreted because they might appear to be quite confusing. Recalling his very first equation, Equation (Hamilton 1), the terms ***scalar*** and ***vector***,

$$Q = \underbrace{w}_{\text{scalar}} + \underbrace{ix + jy + kz}_{\text{vector}}, \tag{1}$$

are obviously quite natural, since w is a number, $x, y, z \in \mathbb{R}$, are coordinates and i, j, k the axes of a Cartesian (rectangular) coordinate system: $SQ = w$, $VQ = ix + jy + kz$.

Very logical appear also the definitions in Equatios (Hamilton 22) and (Hamilton 25) as they follow immediately from the 'parametrization' in Equation (Hamilton 6)

$$\begin{aligned}
(TQ)^2 &= (SQ)^2 - (VQ)^2 && \text{(Hamilton 22)}\\
&= w^2 - [(ix + jy + kz)(ix + jy + kz)]\\
&= w^2 + x^2 + y^2 + z^2,\\
TQ &= (SQ + VQ)(SQ - VQ) && \text{(Hamilton 25)}\\
&= (w + ix + jy + kz)(w - ix - jy - kz)\\
&= w^2 + x^2 + y^2 + z^2 - wix - wjy - wkz.
\end{aligned}$$

In particular, for $w = 0$ the following relations apply

$$\begin{aligned}
&w = 0 : (TQ)^2 = TQ,\\
&w = 0,\ x^2 + y^2 + z^2 = 1 : TQ = 1,
\end{aligned}$$

from which one can read off that indeed the set $\mathcal{A}(w = 0)$,

$$\begin{aligned}
\mathcal{A}(w = 0) = \{&\pm 1, \pm(ix + jy + kz), \pm(ix + jy + kz)(ix + jy + kz) \mid x, y, z \in \mathbb{R};\\
&i^2 = j^2 = k^2 = -1;\ ij = k,\ ji = -k,\ i \to j \to k\},
\end{aligned} \tag{2}$$

forms an algebra, since addition and multiplication of the elements are well defined and the associative and distributive law are valid.

Viewed in terms of the original definition $Q = w + ix + jy + kz$, the relations in Equations (Hamilton 20) and (Hamilton 21), however, seem to be slightly strange, namely as if derived 'simply' by dividing by Q. The specification *detaching the characteristics of operation from the signs of the operands* does not really help in this case to understand Hamilton's intentions.

The term *versor* never became common language, since in the definition of U, $U.TQ = 1$, this would imply to show that $U = (TQ)^{-1}$, i.e. once again the difficulty with the inverse operation arises.

7. 'Differential' quaternions

In continuations [5–8], Hamilton dealt essentially with all sorts of geometrical aspects of quaternions. In continuation [9], however, he introduced a new feature based on the key definition in Equation (Hamilton 6).

… if we introduce a new characteristic of operation, $\lhd$, defined with relation to these three symbols ijk, and to the known operation of partial differentiation, performed with respect to three independent but real variables xyz,as follows:

$$\lhd = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}, \tag{Hamilton 26}$$

this new characteristic $\lhd$ will have the negative of its symbolic square expressed by the following formula:

$$-\lhd^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}, \tag{Hamilton 27}$$

of which it is clear that the applications to analytical physics must be extensive in a high degree. [9] In order to explain, why he suggested $\lhd$ instead of ∇, he added a footnote saying that *this* [∇] *more common sign has been so often used with other meanings that it seems desirable to abstain from appropriating it to the new signification here proposed* [9].

The new operator $\lhd$ *presents itself under the form of a quaternion* [9]:

$$\begin{aligned} \lhd\,(it + ju + kv) = {} & i\frac{dt}{dx} + j\frac{du}{dy} + k\frac{dv}{dz} \\ & + i\left(\frac{dv}{dy} - \frac{du}{dz}\right) + j\left(\frac{dt}{dz} - \frac{dv}{dx}\right) + k\left(\frac{du}{dx} - \frac{dt}{dy}\right), \end{aligned} \tag{Hamilton 28}$$

i.e. Equation (Hamilton 28) is nothing but the multiplication of two quaternions,

$$\begin{aligned} (ix + jy + kz)(it + ju + kv) = {} & -(xt + yu + zv) \\ & + i(yv - zu) + j(zt - xu) + k(xu - yt), \end{aligned} \tag{Hamilton 29}$$

with a similar meaning applying to the product of two 'differential quaternions':

$$\left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right)\left(i\frac{d}{dx'} + j\frac{d}{dy'} + k\frac{d}{dz'}\right).$$

8. Intermission III

His series '*On quaternions: or on a new system of imaginaries in algebra*' consists of 17 (!) continuations of his first paper in the Philosophical Magazine in 1844, the last continuation appeared in 1850. Thirteen of these continuations, namely [3,5–8,10–17] deal with all sorts of geometrical aspects of quaternions. This in turn shows, *inter alia*, how much weight Hamilton gave to geometrical interpretations of his quaternions and to geometrical problems in general. The basic difficulty in these geometrical interpretations, at least those reflecting rotations, is that even in the simplest algebra, see Equation (2), there are two unit elements, namely ± 1.

9. Continued fractions or the search for the inverse

Hamilton was of course completely aware of this difficulty. In his second series of papers, *On Continued Fractions in Quaternions,* [18–22], he tried to overcome it by using a continued fraction technique.

It is required to integrate the equation in differences,

$$u_{x+1}(u_x + a) = b, \quad \text{(Hamilton 30)}$$

where x is a variable whole number, but a, b, u are quaternions. Let q_1 and q_2 be any two assumed quaternions; [18], then after a few lines of derivation, he was able to show that

$$\frac{u_x + q_1}{u_x + q_2} = v_x. \quad \text{(Hamilton 31)}$$

And because in no one stage of the foregoing process has the commutative principle of multiplication been employed, the results hold good for quaternions, and admit of interesting interpretations [18].

The continued fraction

$$u_x = \left(\frac{b}{a+}\right)^x c, \quad \text{(Hamilton 32)}$$

where a, b and c are real quaternions and $+$ is meant to specify only positive units i, j, k, which lead to Equation (Hamilton 31) was then re-examined and generalized in [19] and [20] and even geometrically interpreted [21], but did not yield an expression for the inverse of a quaternion.

Let us now consider the continued fraction, he restarted therefore in [22],

$$u_x = \left(\frac{\beta}{\alpha+}\right)^x u_0, \quad \text{(Hamilton 33)}$$

where u_0 and u_x are quaternions, and α, β are two rectangular vectors, connected by the relation,

$$\alpha^4 + 4\beta^2 = 0, \quad \text{(Hamilton 34)}$$

and, as a sufficient exemplification of the question, let it be supposed that α, β have the values

$$\alpha = i - k, \quad \beta = j. \quad \text{(Hamilton 35)}$$

It may easily be shown, by the rules of the present Calculus, that the expression,

$$u_1 = \frac{j}{i - k + u_0} \quad \text{(Hamilton 36)}$$

gives the relations [22]

$$(u_1 - k)^{-1} = k + i(u_0 - k)^{-1}k, \quad \text{(Hamilton 37)}$$

$$(u_{2n+x} - k)^{-1} - (u_x - k)^{-1} = n(k - i). \quad \text{(Hamilton 38)}$$

Although the last three equations offer interesting results, they also show that either with the mathematical tools available in England in the 1849s and 1850s of the nineteenth century, an algebraic definition of the inverse of a quaternion was not possible, or, that Hamilton's original parametrization in Equation (Hamilton 6) had to changed.

10. Polynomials

In the last series of papers, *On some Extension of Quaternions* (1854–1855), Hamilton tried yet another way to deal with the remaining problem and one has to admit he came pretty close to the answer, at least as far as the basic idea was concerned.

Conceive that in the polynomial expressions,

$$P = i_0x_0 + i_1x_1 + \ldots + i_nx_n = \sum ix \quad \text{(Hamilton 39)}$$

$$P' = i_0x_0' + i_1x_1' + \ldots + i_nx_n' = \sum ix' \quad \text{(Hamilton 40)}$$

$$P'' = i_0x_0'' + i_1x_1'' + \ldots + i_nx_n'' = \sum ix''$$

the symbols $x_0 \ldots x_n$, which we shall call the constituents of the polynome P, and in like manner that the constituents $x_0' \ldots x_n'$ of *P' are subject to all the usual rules of algebra; but that the other symbols, $i_0 \ldots i_n$, by which those constituents of each polynome are here symbolically multiplied, are not all subject to all those usual rules: and that, on the contrary, these latter symbols are subject, as a system, to some peculiar laws, of comparison and combination* [23].

… instead of supposing that the symbols i combine thus in general commutatively, among themselves, as factors or as operators, we shall distinguish generally between the two inverted (or opposite) products, ii' and $i'i$, or $i_f i_g$ and $i_g i_f$; and shall conceive that all the $(n+1)^2$ binary products (ii'), including squares ($i^2 = ii$), of the $n+1$ symbols i, are defined as being each equal to a certain given or originally assumed polynome … [23].

Hamilton was indeed close to a general and correct description of quaternions, since he proposed an equation with a polynomial on the rhs and a linear from of the lhs, see Equation (Hamilton 39). He assumed, however, that on both sides there would be the same kind of *peculiar* objects, i.e. that the $i's$ are of the same kind on both sides, see also Equation (Hamilton 40). Even the condition $ii' \neq i'i$ (his original assumption in 1844) was partially correct. No matter, what type of approach he used in the following papers [24–27], the key equation and the key obstacle were contained in one and the same equation, namely in Equation (Hamilton 39).

In the end, Hamilton's problem was solved only quite a few years later by William Kingdon Clifford (1845–1879). To speak loosely, Clifford had the great idea to view Equation (Hamilton 39) in a different way, namely to consider the two sides of that equation separately: Given a polynomial with real coefficients on one side, if equated with a linear form on the other side, what are the properties of the 'coefficients' in the linear form? As is probably well known, this question leads immediately to so-called Clifford algebras.

By the way, a similar situation, very much related to Hamilton's dilemma, haunted quantum mechanics for a while, essentially till Pauli and Dirac discovered a way, how to linearize a square root. [33] Dirac found more or less by trial and error that in order to include Einstein's special theory of relativity into quantum mechanics, the 'coefficients' in the linear form had to be very special 4x4 matrices: the famous Dirac matrices. Clifford algebras were not known to either Pauli nor Dirac.

11. Conclusion

In his very last note [29] on quaternions, 1862, Hamilton again returned to the problem that 'haunted' him for 18 years:

Quite recently, I have discovered that the far more general linear (or distributive) and quaternion function of a quaternion can be inverted, by an analogous process, or that there always exists, for any such function fq, satisfying the condition

$$f(q+q') = fq + fq'$$

*…; and that therefore we may write generally this **Formula of Quaternion Inversion**,*

$$nf^{-1} = n' - n''f + n'''f^2 - f^3 \dots$$

where n, n', n'', n''' *are four scalar constants, depending on the particular composition of the function* fq.

It seems that with this note, Hamilton was closing up his studies on quaternions, *As it was in the Number of the Philosophical Magazine for July 1844 that the first printed publication of the Quaternions occurred …, I have thought that the Editors of the Magazine might perhaps allow me thus to put on record what seems to myself an important addition to the theory, and that they may even allow me to add, in a Postscript to this communication, so much as may convey a distinct conception of the Method which I have pursued.* He died three years later.

During his time and quite a few years more Hamilton's quaternions were praised as one of the most outstanding pieces of mathematics. Maxwell for example stated that *The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of greatest use in all parts of science.* [34]

Nowadays, Hamilton's concept of quaternions is used among other applications e.g. in the theory of rotations in terms of the Euler-Rodrigues parameterization, namely in the form of $(\alpha, \mathbf{n})$, where α is a rotation angle and $\mathbf{n}$ specifies the rotation axis. [30] It is indeed a bit tragic to think that Hamilton's long-lasting effort in dealing with quaternions did not end in the result he ultimately wanted to achieve.

Acknowledgement

I am grateful to Dr S.L. Altmann for a critical reading of the manuscript.

Notes

1. An exceedingly interesting history accompanies the theory of quaternions as was pointed out in the introduction of Ref. [30], in which also a short biography of Hamilton is compiled.
2. Quotations are indicated by italics, original equations by his name.
3. Mostly likely derived from the Latin word 'quaternio' meaning a set of four.
4. Monday, 16 October 1843, accompanied by Lady Hamilton, he was walking past Broome Bridge in Dublin, when suddenly he realized that three imaginary units were needed to solve the problem he had in mind. He abruptly stopped his walk and carved Equation (Hamilton 6) in the stone of the bridge [30].
5. Such a theory has to include many more famous names such as Euler, Rodrigues, etc. and of course generations of group theoreticians.

References

[1] W.R. Hamilton, Philos. Mag. Ser. 3 25 (1844) p.10; Proc. Royal Irish Acad. 2 (1844) p.424.
[2] W.R. Hamilton, Philos. Mag. Ser. 3 25 (1844) p.241.
[3] W.R. Hamilton, Philos. Mag. Ser. 3 26 (1845) p.220.
[4] W.R. Hamilton, Philos. Mag. Ser. 3 29 (1846) p.26.
[5] W.R. Hamilton, Philos. Mag. Ser. 3 29 (1846) p.113.
[6] W.R. Hamilton, Philos. Mag. Ser. 3 29 (1846) p.326.
[7] W.R. Hamilton, Philos. Mag. Ser. 3 30 (1847) p.458.
[8] W.R. Hamilton, Philos. Mag. Ser. 3 31 (1847) p.214.
[9] W.R. Hamilton, Philos. Mag. Ser. 3 31 (1847) p.278.
[10] W.R. Hamilton, Philos. Mag. Ser. 3 31 (1847) p.511.
[11] W.R. Hamilton, Philos. Mag. Ser. 3 32 (1848) p.367.
[12] W.R. Hamilton, Philos. Mag. Ser. 3 33 (1848) p.58.
[13] W.R. Hamilton, Philos. Mag. Ser. 3 34 (1849) p.294.
[14] W.R. Hamilton, Philos. Mag. Ser. 3 34 (1849) p.340.
[15] W.R. Hamilton, Philos. Mag. Ser. 3 34 (1849) p.425.
[16] W.R. Hamilton, Philos. Mag. Ser. 3 35 (1849) p.133.
[17] W.R. Hamilton, Philos. Mag. Ser. 3 36 (1850) p.305.
[18] W.R. Hamilton, Philos. Mag. Ser. 4 3 (1852) p.371.
[19] W.R. Hamilton, Philos. Mag. Ser. 4 4 (1852) p.303.
[20] W.R. Hamilton, Philos. Mag. Ser. 4 5 (1852) p.321.
[21] W.R. Hamilton, Philos. Mag. Ser. 4 5 (1853) p.117.
[22] W.R. Hamilton, Philos. Mag. Ser. 4 5 (1853) p.236.
[23] W.R. Hamilton, Philos. Mag. Ser. 4 7 (1854) p.492.
[24] W.R. Hamilton, Philos. Mag. Ser. 4 8 (1854) p.125.
[25] W.R. Hamilton, Philos. Mag. Ser. 4 8 (1854) p.261.
[26] W.R. Hamilton, Philos. Mag. Ser. 4 9 (1855) p.46.
[27] W.R. Hamilton, Philos. Mag. Ser. 4 9 (1855) p.280.
[28] W.R. Hamilton, Philos. Mag. Ser. 4 12 (1856) p.446.
[29] W.R. Hamilton, Philos. Mag. Ser. 4 24 (1862) p.127.
[30] S.L. Altmann, *Rotations, Quaternions, and Double Groups*, Oxford University Press, Oxford, 1986; see in particular: S.L. Altmann, Math. Mag. 62 (1989) p.291.
[31] A. Cayley, Philos. Mag. Ser. 3 26 (1845) p.141.
[32] A. Cayley, Philos. Mag. Ser. 3 33 (1848) p.196.
[33] P. Weinberger, Philos. Mag. 88 (2008) p.2585.
[34] J.C. Maxwell, Proc. London Math. Soc. 3 (1869) p.226.