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COMMENTARY

Laplace and the era of differential equations

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Between about 1790 and 1850 French mathematicians dominated not only mathematics, but also all other sciences. The belief that a particular physical phenomenon has to correspond to a single differential equation originates from the enormous influence Laplace and his contemporary compatriots had in all European learned circles. It will be shown that at the beginning of the nineteenth century Newton's "fluxionary calculus" finally gave way to a French-type notation of handling differential equations. A heated dispute in the *Philosophical Magazine* between Challis, Airy and Stokes, all three of them famous Cambridge professors of mathematics, then serves to illustrate the era of differential equations. A remark about Schrödinger and his equation for the hydrogen atom finally will lead back to present times.

Keyword: history of science

1. Introduction

The turn of the eighteenth to the nineteenth century was *the* epoch of French mathematics. The names of all the famous mathematicians of that time such as Jean le Rond d'Alembert (1717–1783), Alexandre-Théophile Vandermonde (1735–1796), Joseph-Louis Lagrange (1736–1813), Pierre-Simon Laplace (1749–1827), Jean Baptiste Joseph Fourier (1768–1830), Adrien-Marie Legendre (1752–1833), Siméon Poisson (1781–1840), Augustin-Louis Cauchy (1789–1857), and Évariste Galois (1811–1832) sound very familiar to us *a posteriori*. It is indeed amazing to realize that perhaps with the exception of d'Alembert, who might be considered as the *spiritus rector*[1] of that splendid period, and Vandermonde, all of them were contemporaries. But, no other intellectual influenced European science and thinking in the first half of the nineteenth century more than Laplace, who, by the way, became a count of the First French Empire in 1806 and – after the Bourbon Restoration – was named a marquis in 1817. Or, using Fourier's words on the occasion of the obituary read at a public meeting of the Royal Academy

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of Science:

[370]

LIX.—*Historical Eloge of the Marquis De Laplace**.—By M. Le Baron Fourier.

THE name of Laplace has been heard in every part of the world where the sciences are honoured; but his memory could not receive a more worthy homage than the unanimous tribute of the admiration and sorrow of that illustrious body who shared in his labours and in his glory. He consecrated his life to the study of the grandest objects which can occupy the human mind.

The wonders of the heavens,—the lofty questions of natural philosophy,—the ingenious and profound combinations of mathematical analysis,—all the laws of the universe have been presented to his thoughts during more than sixty years, and his efforts have been crowned with immortal discoveries.

From Fourier's obituary "*pronounced at the public sitting of the Royal Academy of Sciences on the 15th June 1829*" [1]

2. British mathematics between 1800 and 1850

It is quite revealing to follow up the changes in attitude in British mathematics (and therefore also in North America) that took place due to Laplace's enormous influence. Clearly Newton, the inventor of differential calculus[2] or of *fluxionary* calculus as it was termed then, had primarily the description of motions in mind, from which he also derived his terminology and which still can be found in some old-fashioned textbooks on mechanics, namely the use of dotted and doubled dotted variables such as, e.g. $\dot{x}$ and $\dot{x}\dot{y}$ instead of dx and d$x\,$dy. In scanning in the *Philosophical Magazine Archives* the years from around 1800 to about 1850, these changes – and partially also the (then) existing admiration for French mathematics, in particular for Laplace – become apparent.

2.1. *The end of fluxionary calculus*

As early as in 1801, Dickson [2] insisted in a longish article that differentials in fact arise from applying geometrical concepts rather than from Newton's laws of motion (*fluxionary* calculus) and summarized rules for finding differentials. He even didn't hesitate to include a rather impolite footnote accusing more or less contemporary British mathematicians of being stubborn:

> "*Here one is almost tempted to ask, Whether the ingenious author considers the British mathematicians as mere Differentials? For they have never agreed to use the notation he mentions; but, instead of* dx, d$x\,$dy, &*c write, with Newton, the immortal inventor of fluxions*, $\dot{x}$, $\dot{x}\dot{y}$ &*c. The d's only serve to embarrass the combinations, which should be expressed with utmost clearness. W.D.*"

Footnote on p. 40 of his article [2].

In the course of his treatise, however, he occasionally has to recall *fluxionary* language to make sure that he is not misunderstood.

> "*From what has be said, it appears that the differential here required is, that is we have* $y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}x\,\mathrm{d}y$"
>
> Footnote indicated by † on pp. 44–45: "*As the ingenious author has touched no farther on the practise than seemed necessary to elucidate his theory. I shall endeavour to show ... how to find the differentials, or, which is the same thing, the fluxions of products, powers, roots, and fractions.*
>
> *To find the fluxions of products, such as* xy, xyz, &c, *Example* 1. $(x + \dot{x}) \times (y + \dot{y}) = xy + x\dot{y} + y\dot{x} + \dot{x}\dot{y}$. *But, with respect to this equation, I observe, that* $\mathrm{d}x$ *and* $\mathrm{d}y$*, being infinitely small, in comparison with x and y, the last. But, for reasons which the author gives,* $\dot{x}\dot{y}$ (*in his notation* $\mathrm{d}x\,\mathrm{d}y$) *may be rejected, and so the fluxion of* xy *is* $x\dot{y} + y\dot{x}$." [2]

It actually turned out that *fluxions* were rather long-lived. In 1810 an article with the title "*On prime and ultimate ratios; with their application to the first principle of the fluxionary calculus*" appeared [3] in which again the differential of xy was formulated in terms of fluxions.

Before going ahead a remark has to be made with respect to the convention for mathematical symbols. One has to be aware of the fact that essentially only because of Adam Ries[3] (1492–1559), i.e. from the sixteenth century on, Arab numerals started to become customary for the purposes of counting and that operational signs such as a plus (+) for addition or the letter x for multiplication were by no means universally accepted even at the beginning of the nineteenth century. In 1815, for example, the following rules for algebraic multiplication were published [4]:

> " *a* × *b evidently means that a is to be taken b times. a* + *b* × *c, that the sum of a and b is to be taken c times: but as a and b cannot in this state be really added together, we can only say that when they can, a is to be taken c times, and b is to be taken c times, and the mode of writing this direction shortly is ca* + *cb.*
>
> *That* + × − *is minus, admits of an equally clear demonstration or rather explanation. We must first remember that no quantity simply considered can be*− *minus. It must be compared with some other quantity either greater than itself or of opposite direction. In the common operations of algebra it is only used in the former sense, and a* − *b merely means that b being less than a, it is when it can be done, to be subtracted from a; a* − *b therefor really means the difference between a and b.*" [4]

Quite clearly in the "present day understanding" $ca + cb = c(a + b)$, which, however, is only a "practical" convention, since Gödl refuted rigorously Wittgenstein's command that in logical expressions the use of brackets is compulsory. The use of set theory symbols such as $\cap$ or $\subset$, for example, only became customary in the twentieth century. This short deviation is to remind us that also in mathematics – like in all other sciences – certain topics can play an important role for a certain period of time. One of these periods was the era of differential equations.

2.2. *The Laplace equation*

The British scientific community was fascinated by the Laplace equation [5,6], which in fact, according to some historians, ought to be attributed rather to Laplace's older colleague Lagrange. Laplace's *Mechanique Céleste* seems to have been *the* favorite book of the learned British public and societies. In a contribution [7] (1813) to the *Philosophical Magazine* entitled "*Derivation of one of the Equations in Laplace's Mechanique Céleste*", for example, a *very* detailed derivation of the Laplace equation in polar coordinates [8] is given, of course not using fluxionary calculus. This contribution obviously increased further the public interest in the Mechanique Céleste, since in the same year a reader [9] of that journal asked for help to understand certain sections of Laplace's book. Later on, even additional proofs for certain theorems of Laplace and Lagrange appeared [10].

It would be incorrect to claim that mathematics in the English speaking world was exclusively devoted to understanding the ideas of Laplace and all the other French mathematicians: publications dealing with practical aspects of Taylor's series [11] or the method of least squares [12], or (quite a few) with algebraic descriptions of geometrical problems appeared on a more or less regular basis. However, it was not only the question of the origin of the universe that the title of Laplace's book seemed to address and that at all times attracted the attention of many, but also the firm belief associated with it, namely that astronomy and physics *per se* can be described by the proper knowledge of the corresponding differential equations. In this sense, the first half of the nineteenth century can be viewed as an epoch in mathematics and physics dominated by differential equations.

In addition to this belief the practicability of some of Laplace's new concepts obviously impressed the British scientific community quite a bit as can be judged for example from an application of his probability function [13] to geodesic operations discussed by him [14] in full detail in 1821.

2.3. *Stokes and his equation*

Probably the best example to prove the claim of the existence of an era of differential equations in (mathematical) physics is to be found in a long lasting intellectual controversy between three Cambridge scientists, namely James Challis[4] (1803–1882, Plumian Professor of Astronomy and Experimental Philosophy), George Airy[5] (1801–1892, Lucasian Professor of Mathematics and Royal Astronomer) and George Stokes[6] (1819–1903, Lucasian Professor of Mathematics) fought in the *Philosophical Magazine*. They first battled over the motion of fluids, then over certain optical properties and the propagation of sound. In the following the motion of fluids shall serve as an example.

Challis first started out in 1829 [15] to suggest an integration for a differential equation listed by Poisson:

> "*The theoretical investigation of the laws of motion of incompressible fluids, conducted in the most general manner possible, leads to the equations*

$$\frac{p}{\rho} = V - \frac{\mathrm{d}\Phi}{\mathrm{d}t} - \frac{1}{2}\left(u^2 + v^2 + w^2\right) \tag{1}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2} + \frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} = 0 \tag{2}$$

$$u = \frac{d\Phi}{dx}, \quad v = \frac{d\Phi}{dy}, \quad w = \frac{d\Phi}{dz} \tag{3}$$

(*Poisson, Traitè de Mécanique, tom. ii p.* 486)

ρ is the density of the fluid, p the pressure at any point, the co-ordinates of which are x, y, z; u, v, w are the velocities in the directions x, y, z, respectively; $dV = X\,dx + Y\,dy + Z\,dz$, *X, Y, Z, being the accelerative forces impressed at the point*; *and* Φ *is a function of x, y, z, and t, such that*

$$(d\Phi) = u\,dx + v\,dy + w\,dz. \tag{4}$$

Consequently the above equations apply only to cases in which $u\,dx + v\,dy + w\,dz$ *is a complete differential of x, y, and z*." [15]

It should be noted that Challis's Equation (2) is in fact the Laplace equation. The question of whether or not $d\Phi$ is indeed a complete differential will turn out to be the main cause of that heated dispute.

In 1840 he published [16,17] yet another differential equation that however was immediately disputed. The speed at which the dispute was carried out in the *Philosophical Magazine* is quite astonishing, the wording by the way being rather frostily polite.

"*If I am correct in the view I have taken of the connexion of these steps of the process, I conceive that the investigation must be considered faulty*", **Airy** [18], December 9, 1840, commenting on Challis's paper [17] from November 16, 1840.

"*Allow me ... to express my thanks to Mr. Airy for calling my attention ... to a step in my solution of the problem of the resistance to a sphere vibrating in an elastic medium, which I had left unexplained*", **Challis** [19], January 18, 1841, in response to Airy's critique [18].

"*Without professing to have examined the equation minutely, I can only say at present that the three equations appear to me to be inconsistent, and therefore I consider that the motion assumed by Professor Challis is impossible ...*", **Airy** [20], March 22, 1841 in answering Challis's reply [19].

"*... to avoid fruitlessly prolonging the present discussion, I willingly express the satisfaction it has given me, that the Astronomer Royal ... should have thought this subject worthy of so much attention; and though I do not see reason for changing my first views, I acknowledge that they have become clearer on several points by the remarks which this discussion has elicited*", **Challis** [21], June 19, 1841, replying to Airy's comment [20].

"*... As to the omission of certain terms in the process of differentiation, I confess that I am surprised ...*
I will only, in conclusion, express my regret at finding myself compelled to place myself so distinctly in opposition to my excellent friend Professor Challis ... I have only to add, that nothing could be further from my intention than to give a personal character to this controversy, and that I trust no expression has escaped me which will bear such an interpretation", **Airy** [22], July 10, 1841, closing his dispute with Challis.

In the meantime Challis started to worry [23] about Equation (4) in his original paper [15] from 1829:

"*No general rule has hitherto been given for determining when it is allowable to assume* $u\,dx + v\,dy + w\,dz$ *to be an exact differential, nor has it been ascertained to what particular circumstance of the motion this analytical condition refers. This must be considered a defect in the mathematical theory of hydrodynamics*." [23]

For this reason, he starts over again by proposing "*a new equation in hydrodynamics*" [24,25] in the absence of extraneous forces,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\frac{\mathrm{d}\cdot\rho u}{\mathrm{d}x}+\frac{\mathrm{d}\cdot\rho v}{\mathrm{d}y}+\frac{\mathrm{d}\cdot\rho w}{\mathrm{d}z}=0 \tag{5}$$

$$(\mathrm{d}P)+\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)\mathrm{d}x+\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}y+\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)\mathrm{d}z=0 \tag{6}$$

where – as before – ρ is the density and $p=k\rho$ the pressure, k being a constant; u, v, w are the velocities in the direction of the coordinates x, y and z, and $P=k\,\mathrm{Nap.log}\,\rho$ with Nap.log referring to John Napier's[7] (1550–1617) definition[8] of the logarithm. In essence, however, his theoretical treatment condenses once again to the question of whether or not $u\,\mathrm{d}x+v\,\mathrm{d}y+w\,\mathrm{d}z$ can be made integrable. Later on, in a further paper [26] on this topic he claims that if $u\,\mathrm{d}x+v\,\mathrm{d}y+w\,\mathrm{d}z$ is a complete differential then the motion of the fluid is rectilinear. This statement provoked an almost immediate reply by another well-known Cambridge mathematician, namely Stokes:

> "*In the August Number of this Magazine (p. 101), Professor Challis has written an article, of which the object is to prove that, in all cases of fluid motion in which* $u\,\mathrm{d}x+v\,\mathrm{d}y+w\,\mathrm{d}z$ *is an exact differential, the motion is rectilinear. The importance of this question may apologize for these remarks, since, if the reasoning in that article be correct, it will affect the validity of much that has been written on the subject. It appears to me however that Professor Challis has made an assumption which is not allowable, and consequently the conclusion founded on it is not allowable either*", **Stokes's** [27] comment on Challis's rectilinear motion.

> "*Mr. Stokes has brought forward four arguments against a new theorem in hydrodynamics which I have advanced, viz. that fluid motion is rectilinear whenever* $u\,\mathrm{d}x+v\,\mathrm{d}y+w\,\mathrm{d}z$ *is an exact differential …*
> *In the first argument it is contended that my demonstration … takes no account of the curvature of the lines of motion. I admit the validity of this objection. … I have given proves only … if the surfaces of displacement are surfaces of equal velocity …* " **Challis's** [28] reply to Stokes.

> "*I cannot see where Professor Challis conceives it to be proved that* dx, dy, dz *are independent of time, in the equation* $u\,\mathrm{d}x+v\,\mathrm{d}y+w\,\mathrm{d}z=0$, *which is the differential equation to a surface of displacement, supposing the first side an exact differential.*" **Stokes** [29] answering Challis's reply.

Looking back, one has to say that although Challis seems to have only circled around an adequate description of hydrodynamics, his contributions most likely have added to the present day formulation of the Navier–Stokes equation(s), namely a system of nonlinear partial differential equation(s)

$$\rho\left(\frac{\partial v}{\partial t}+v\cdot\nabla v\right)=-\nabla p+\nabla\cdot\mathbb{T}+f \tag{7}$$

where v is the flow velocity, ρ the fluid density, p the pressure, $\mathbb{T}$ the (deviatoric) stress tensor and f represents the body forces (per unit volume) acting on the fluid. In terms of appropriate models these equations are extremely useful for the prediction of weather, ocean currents, or, e.g. for the air flow around airplane wings. Even now, in the twenty-first century it seems that it cannot be proven in purely mathematical terms that in three dimensions solutions of this equation always exist or that they would not contain singularities (perhaps somebody should tell Challis, Airy and Stokes). The Navier–Stokes equations surely belong to the most prominent equations in classical physics; their origin, however, has to be traced back to the overwhelming influence French mathematics and mathematical physics had in the nineteenth century. Laplace is just a synonym for this age in science.

3. Nineteenth century French mathematics and present day physics

The main idea in Schrödinger's famous series of papers (1926) [30–33], "*Quantisierung als Eigenwertproblem*", for example, is based on putting proper boundary conditions on already well-known differential equations. Recall for a moment his time-independent equation for a central field $V(r) = -Ze^2/r$, $r = |\mathbf{r}|$,

$$\left(-\frac{\hbar}{2\mu}\nabla^2 + V(r)\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}),$$

which results from a separation of the motion of the nucleus from that of an electron in terms of **Lagrange** parameters (μ is the reduced mass). As is probably well-known, by applying the **Laplace** operator in polar coordinates [8] a separation of independent motions leads to **Laguerre** polynomials[9] for the "radial motion" and to **Legendre** polynomials as part of the angular motion. In this context it is perhaps interesting to note that in his first paper, for the differential equation for the radial motion, he needed a Laplace transformation [34] to argue, since obviously neither he nor Weyl[10] were aware of **Laguerre** polynomials. In his third paper, however, he even discusses properties of these polynomials citing "*Courant–Hilbert*, *Kap. II*, § 11,5. *S*. 78, *Gleichung* (72)" as a footnote, namely by making use of that book [35] that only appeared two years earlier (1924).

Besides Quantum Mechanics, nineteenth century French mathematics still seems to dominate essential parts of physics: spectroscopy, e.g. makes extensive use of Fourier analysis or of distribution functions, and quite a few of the numerical recipes in computational physics are related to mathematical ideas expressed in the first half of the nineteenth century. The enormous number of so-called *ab initio* calculations and applications of Density Functional Theory that were published over the last 30 years would not have been possible without the use of the **Poisson** equation.

The early stages of this development can easily be followed in a proper historical (and social) context in the *Philosophical Magazine Archives*, for example in terms of the Navier–Stokes equation(s), as discussed above.

Notes

1. In fact he was the "teacher" of Laplace.
2. Independently of the German philosopher and mathematician (Gottfried Wilhelm) Leibniz (1646–1716).
3. http://en.wikipedia.org/wiki/Adam_Ries
4. http://en.wikipedia.org/wiki/James_Challis
5. http://en.wikipedia.org/wiki/George_Biddell_Airy
6. http://en.wikipedia.org/wiki/Sir_George_Stokes,_1st_Baronet
7. http://en.wikipedia.org/wiki/John_Napier

8. Nap.log $(10^7) = 0$; Nap.log $y = 10^7.\log_{1/e}(y/10^7)$
9. Edmond Laguerre (1834–1886), http://en.wikipedia.org/wiki/Edmond_Laguerre
10. http://en.wikipedia.org/wiki/Hermann_Weyl

References

[1] M. Le Baron Fourier, Phil. Mag. Series 2 6 (35) (1821) p.370.
[2] W. Dickson, Phil. Mag. Series 1 9 (33) (1801) p.39.
[3] Mr. Marat, Phil. Mag. Series 1 36 (149) (1810) p.186.
[4] H.C. Englefield, Phil. Mag. Series 1 45 (201) (1815) p.15.
[5] The *Laplace equation*,

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = 0$$

is a second-order partial differential equation with f being a real-valued function. If

$$\Delta f = g,$$

where g is also a real-valued function, then the *Laplace equation* is called the *Poisson equation*.
[6] In Cartesian coordinates the *Laplace equation* is given by

$$\Delta f(x, y, z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) = 0,$$

x, y and z being real variables.
[7] T. White, Phil. Mag. Series 1 41 (177) (1813) p.8.
[8] In spherical coordinates the *Laplace equation* reads as

$$\Delta f(r, \theta, \varphi) = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial f(r, \theta, \varphi)}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f(r, \theta, \varphi)}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f(r, \theta, \varphi)}{\partial \varphi^2}\right] = 0.$$

[9] J. Thomson, Phil. Mag. Series 1 41 (181) (1813) p.357.
[10] J. Gordon, Phil. Mag. Series 2 9 (52) (1831) p.253.
[11] Phil. Mag. Series 1 29 (115) (1807) p.211.
[12] J. Ivory, Phil. Mag. Series 1 68 (341) (1826) p.161.
[13] The *Laplace probability density function* is defined by

$$f(x; \alpha, b) = \frac{1}{2b}\exp\left(-\frac{|x - \alpha|}{b}\right),$$

where x is a random variable, $b > 0$ and α is a so-called "location parameter", $f(x; 0, 1) = \exp(-x)/2$.
[14] Count De Laplace, Phil. Mag. Series 1 58 (280) (1821) p.133.
[15] J. Challis, Phil. Mag. Series 2 6 (32) (1829) p.123.
[16] J. Challis, Phil. Mag. Series 3 3 (15) (1833) p.185.
[17] J. Challis, Phil. Mag. Series 3 17 (112) (1840) p.462.
[18] G.B. Airy, Phil. Mag. Series 3 17 (113) (1840) p.481.
[19] J. Challis, Phil. Mag. Series 3 18 (115) (1841) p.130.
[20] G.B. Airy, Phil. Mag. Series 3 18 (118) (1841) p.321.
[21] J. Challis, Phil. Mag. Series 3 19 (121) (1841) p.63.
[22] G.B. Airy, Phil. Mag. Series 3 19 (122) (1841) p.143.
[23] J. Challis, Phil. Mag. Series 3 18 (119) (1841) p.477.

[24] J. Challis, Phil. Mag. Series 3 20 (129) (1842) p.84.
[25] J. Challis, Phil. Mag. Series 3 20 (131) (1842) p.281.
[26] J. Challis, Phil. Mag. Series 3 21 (136) (1842) p.101.
[27] G.G. Stokes, Phil. Mag. Series 3 21 (138) (1842) p.297.
[28] J. Challis, Phil. Mag. Series 3 21 (140) (1842) p.423.
[29] G.G. Stokes, Phil. Mag. Series 3 22 (142) (1843) p.55.
[30] E. Schrödinger, Ann. Phys. 79 (1926) p.361.
[31] E. Schrödinger, Ann. Phys. 79 (1926) p.489.
[32] E. Schrödinger, Ann. Phys. 80 (1926) p.437.
[33] E. Schrödinger, Ann. Phys. 81 (1926) p.109.
[34] A *Laplace transformation* is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t, \quad t \geq 0 \in \mathbb{R}, \mathrm{s} \in \mathbb{C},$$

where $f(t)$ is a function locally integrable on $[0, \infty)$. In short: a *Laplace transformation* converts a function $f(t)$ with a real argument t into a function $F(s)$ with a complex argument s.
[35] D. Hilbert and R. Courant, *Methoden der mathematischen Physik*, *Methods of Mathematical Physics*, John Wiley, New York, 1991, first published in 1924; present English edition.